Pergamon

0021–8928(95)00011-9

J. Appl. Maths Mechs, Vol. 59, No. 1, pp. 93–98, 1995

0021–8928/95 $24.00+0.00

A HAMILTONIAN APPROACH TO THE INVESTIGATION OF THE POTENTIAL MOTIONS OF AN IDEAL FLUID†

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(*Received 23 November 1993*)

Canonical variables are obtained for the equations of the potential motion of an ideal fluid in arbitrary curvilinear Euler and Lagrange systems of coordinates. The boundary conditions are written in Hamiltonian form with a Hamiltonian equal to the total energy of the fluid. The main purpose of this paper is to develop a Hamiltonian approach to the investigation of non-linear wave processes in volumes of an ideal fluid of non-trivial geometrical shape.

Evolutional wave problems are usually reduced to finding and analysing the solutions of "truncated" shallow-water, Korteweg–de Vries equations, the non-linear Schrödinger equation, etc. [1–4]. Methods of solving these equations are well developed [1–5]. The properties of the solutions obtained depend very much on the dispersion relation and the coefficients of the non-linear terms.

Finding the coefficients of the non-linear terms of the "truncated" equations involves carrying out lengthy asymptotic expansions in the initial system of equations [2, 4, 6, 7]. To simplify the expansions, the Luke variational principle [7] or the Hamiltonian approach [3, 7–9] have been used. When using these methods the asymptotic expansions are only carried out in a single functional (the Lagrangian or Hamiltonian), while the "truncated" equations are automatically found from the variational principle, which considerably reduces the amount of algebraic calculations required.

Lagrange's equations [11, 12] are used to investigate the potential motions of a fluid with a deformable boundary. One can choose the normal displacements of the boundary as the generalized coordinates. The generalized forces are then the pressure forces acting on the boundary of the fluid. This approach is equivalent to the Hamiltonian formalism in a Lagrangian system of coordinates considered in this paper.

Canonical variables have been obtained in [8] for the equations of potential motions of an infinitely deep layer of an ideal fluid with a free surface, written in a Cartesian Euler system of coordinates. Canonical variables were obtained in [9] for the potential motions of a multilayer ideal fluid in a Cartesian system of coordinates and the corresponding Hamiltonian was constructed.

1. The calculation of the vortex-free motion of a volume V of an ideal incompressible fluid reduces to solving Laplace's equation with boundary conditions which have the following form in invariant geometrical notation

$$g^{ij}\nabla_i \frac{\partial\varphi}{\partial x^j} = 0, \quad x^i \in V, \quad i = 1,2,3 \tag{1.1}$$

$$d\Gamma^f/dt = 0, \quad F + p^f/\rho = 0, \quad x^i \in \partial V^f \tag{1.2}$$

$$d\Gamma^r/dt = 0, \quad x^i \in \partial V^r \tag{1.3}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + g^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial}{\partial x^j}, \quad f = \frac{\partial\varphi}{\partial t} + \frac{1}{2}g^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j} + U$$

Here $\varphi(x^i, t)$ is the velocity potential in an arbitrary curvilinear Euler system of coordinates x^i, g^{ij} is the metric tensor of Euclidean space, ∇_i is the symbol of covariant differentiation, $U(x^i, t)$ is the potential of the external mass forces, $p^f (x^i \in \partial V^f, t)$ is the external pressure acting on the free surface of the fluid, ∂V^f, ρ = const is the density of the fluid. We will assume that the boundary ∂V of the volume V consists of two parts: ∂V^f and ∂V^r. The surface ∂V^f is free and is described by the equation $\Gamma^f(x^i, t) = 0$, and the surface ∂V^r is a solid movable wall and its motion is defined by the equation $\Gamma^r(x^i, t) = 0$.

The solution of Laplace's equation in a volume V is uniquely defined if we know the values of the potential φ^f at the boundary ∂V^f, and the function Γ^f, and condition (1.3) is satisfied. Hence, the total

†*Prikl. Mat. Mekh.* Vol. 59, No. 1, pp. 102–108, 1995.

energy of the fluid E, the potential φ and the pressure P are functionals of φ^f and Γ^f. The boundary conditions (1.2) determine the evolution of the boundary value of the potential φ^f and the form of the free boundary. The pressure in the whole volume of the fluid is given, after finding φ, by the formula

$$p = -\rho E \tag{1.4}$$

We will assume that the Euler system of coordinates can be chosen so that the equality $\Gamma^{fr} = h^{fr}(x^1, x^2, t) - x^3$ is satisfied in a finite time, the coordinate lines x^3 have no tangents to the surface ∂V, and the range of variation of x^1 and x^2 is independent of time. Further, obtain the canonical variables we will use coordinate transformations which do not change the direction of the basis vector e_3, tangential to the coordinate line x^3.

2. The total energy E is equal to the sum of the kinetic and potential energies of the fluid and is given by the formula

$$E = T + \Pi, \quad T = \frac{1}{2}\int_V g^{ij}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}dV, \quad \Pi = \int_V U dV$$

The change in E during the motion is equal to the work of the pressure forces on the surfaces ∂V^f and ∂V^r

$$\delta E = -\int_{\partial V^f} p^f \frac{\delta h^f ds}{\rho|\nabla\Gamma^f|} - \int_{\partial V^r} p^r \frac{\delta h^r ds}{\rho|\nabla\Gamma^r|} \tag{2.1}$$

$$|\nabla\Gamma^{f,r}|^2 = g^{ij}\frac{\partial\Gamma^{f,r}}{\partial x^i}\frac{\partial\Gamma^{f,r}}{\partial x^j}, \quad \partial\Gamma^{f,r} = \delta h^{f,r}$$

(p^r is the pressure of the fluid on the solid wall). Note that the quantity p^f in (1.5) is specified, and p^r is found from (1.4).

We will calculate δE assuming that $\delta h^f = \delta h^r$ on the boundary line between ∂V^f and ∂V^r. The variation of the values $\Phi^{f,r}(x^1, x^2, t)$ of the arbitrary function $\Phi(x^i, t)$ on ∂V is related to its variation at a fixed point of space as follows:

$$\delta\Phi^{f,r} = (\delta\Phi + \nabla_3\Phi\delta h)^{f,r}, \quad \Phi^{f,r} = \Phi(x^i \in \partial V^{f,r})$$

Using the Gauss–Ostrogradskii theorem and the formula for integration by parts, we obtain

$$\delta E = -\frac{1}{2}(I_\Gamma^f + I_\varphi^f + I_\Gamma^r + I_\varphi^r) + \int_{S^f}[U\delta h\sqrt{g}]^f dx^1dx^2 + \int_{S^r}[U\delta h\sqrt{g}]^r dx^1dx^2 \tag{2.2}$$

$$I_\Gamma^{f,r} = \int_{S^{f,r}}\left[\nabla_3\left(\frac{\partial\Gamma^{f,r}}{\partial x^i}\frac{\partial\varphi}{\partial x^j}\varphi\right)g^{ij}\sqrt{g} - (1-\delta_{i3})\left[\frac{\partial}{\partial x^i}\left(\varphi\frac{\partial\varphi}{\partial x^j}g^{ij}\sqrt{g}\right) + \right.\right.$$

$$\left.\left. + \frac{\partial}{\partial x^3}\left(\varphi\frac{\partial\varphi}{\partial x^j}g^{ij}\sqrt{g}\right)\frac{\partial\Gamma^{f,r}}{\partial x^i}\right]\right]^{f,r}\delta h^{f,r}dx^1dx^2$$

$$I_\varphi^{f,r} = \int_{S^{f,r}}\left[g^{ij}\delta\left(\varphi\frac{\partial\varphi}{\partial x^i}\right)\frac{\partial\Gamma^{f,r}}{\partial x^j}\sqrt{g}\right]^{f,r}dx^1dx^2$$

The regions $S^{f,r}$ are the projections of the surfaces $\partial V^{f,r}$ onto the plane (x^1, x^2).

We will convert the sum $I_\varphi^f + I_\varphi^r$ assuming that $\partial\varphi$ satisfies Laplace's equation

$$I_\varphi^f + I_\varphi^r = \int_V \nabla^i\left(\delta\varphi\frac{\partial\varphi}{\partial x^i} + \varphi\frac{\partial\delta\varphi}{\partial x^i}\right)dV = 2\int_V \nabla^i\left(\delta\varphi\frac{\partial\varphi}{\partial x^i}\right)dV =$$

$$= 2\int_{S^f}\left[\frac{\partial\varphi}{\partial x^i}\frac{\partial\Gamma^f}{\partial x^j}g^{ij}\sqrt{g}\delta\varphi\right]^f dx^1dx^2 + 2\int_{S^r}\left[\frac{\partial\varphi}{\partial x^i}\frac{\partial\Gamma^r}{\partial x^j}g^{ij}\sqrt{g}\delta\varphi\right]^r dx^1dx^2$$

Differentiating under the sign of the integrals $I_\Gamma^{f,r}$ we obtain

$$I_\Gamma^{f,r} = \int\limits_{S^{f,r}} \left[\frac{\partial\varphi}{\partial x^j}\left(-\frac{\partial\varphi}{\partial x^i}\sqrt{g} + \varphi\frac{\partial\Gamma^{f,r}}{\partial x^i}\frac{\partial\sqrt{g}}{\partial x^3}\right) g^{ij}\right]^{f,r} \delta h^{f,r} dx^1 dx^2$$

We make the following change of coordinates

$$y^1 = x^1, \quad y^2 = x^2, \quad y^3 = \int\limits_h^{x^3} \sqrt{g}dx^3, \quad h = h(x^1, x^2), \quad (x^1, x^2) \in S^{f,r}$$

(h is an arbitrary function of x^1, x^2), without changing the direction of the vectors of the basis ε_3. The Jacobian of this change is $\sqrt{g}$. Hence, the quantity $\sqrt{g}$ in the new system of coordinates is equal to unity, and we can write

$$\frac{\delta E}{\delta\varphi^f} = -\frac{\partial\varphi}{\partial y^i}\frac{\partial\Gamma^f}{\partial y^j}g^{ij}, \quad \frac{\delta E}{\delta\eta} = \frac{1}{2}\frac{\partial\varphi}{\partial y^i}\frac{\partial\varphi}{\partial y^j}g^{ij} - \frac{\partial\varphi}{\partial y^i}\frac{\partial\Gamma^f}{\partial y^j}\frac{\partial\varphi}{\partial y^3}g^{ij}$$

$$y^i \in \partial V^f, \quad \eta = \int\limits_h^{h^f} \sqrt{g}dx^3$$

The variation $(\delta\varphi)^r = (\partial\varphi/\partial t)^r\delta t$ depends functionally on $\delta\varphi^f$, δh^f. Hence, it follows from (1.4) that the sum of the integrals over the surface S^r in (2.3) is equal to

$$-\int\limits_{\partial V^r} p^r \frac{\delta h^r ds}{\rho|\nabla\Gamma^r|}$$

Hence, we have the following assertion.

Assertion. The evolution of the free boundary of a volume of ideal fluid, defined by the equation $x^3 = h(x^1, x^2, t)$, and the value of the velocity potential φ^f on it are determined in an arbitrary curvilinear Euler system of coordinate with metric g^{ij} by the Hamilton canonical equations

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta(\varphi^f)}, \quad \frac{\partial\varphi^f}{dt} = -\frac{\delta H}{\delta\eta} \tag{2.3}$$

and the Hamiltonian is defined by the formula

$$H = E + \int\limits_{\partial V^f} p^f \frac{h^f ds}{\rho|\nabla\Gamma^f|} - \int\limits_{\partial V^r} \frac{\partial h^r}{\partial t}\varphi^r \frac{ds}{|\nabla\Gamma^r|} \tag{2.4}$$

When calculating the Hamiltonian it is assumed that the value of φ over the whole volume of motion V is also related to the canonical variable φ^f in the same way as the solution of Laplace's equation (1.1) with boundary condition (1.3) is related to the boundary value of the potential φ on ∂V^f.

The canonical variable η has a clear geometrical meaning. Assuming $h = h^f(x^1, x^2, t = 0)$, we find that the quantity $\eta(x^1, x^2, t)dx^1dx^2$ is equal to the elementary volume ∂V traversed by an elementary area lying on the free surface of the fluid in the direction of the x^3 axis in a time t.

As an example we will consider the canonical variables for motions which can be conveniently investigated in cylindrical or spherical systems of coordinates. In the first case we have

$$x^1 = r, \quad x^2 = \phi, \quad x^3 = z, \quad g_{11} = 1, \quad g_{22} = r, \quad g_{33} = 1$$

$$g_{ij} \neq 0 \text{ when } i = j, \quad \sqrt{g} = r$$

We will assume that the free surface of the liquid ∂V^f is described by the equation $r = h^f(\phi, z, t)$. In this case the canonical variables are the values of the potential φ^f and ∂V^f and

$$\eta = \int_0^{h^f} r dr = \frac{1}{2}(h^f)^2$$

For a spherical system of coordinates we obtain

$$x^1 = r, \quad x^2 = \theta \text{ (polar angle)}, \qquad x^3 = \lambda \text{ (length)}$$

$$g_{11} = 1, \quad g_{22} = r, \quad g_{33} = r^2 \sin^2\theta, \quad g_{ij} = 0 \quad \text{for} \quad i \neq j, \qquad \sqrt{g} = r^2$$

If the free surface ∂V^f of the fluid is described by the equation $r = h^f(\theta, \lambda\, t)$, the canonical variables will be φ^f and

$$\eta = \int_0^{h^f} r^2 dr = \frac{1}{3}(h^f)^3$$

The approach considered can be extended to the investigation of the motion of fluid volumes taking into account non-inertial and non-dissipative surface phenomena on the boundary ∂V^f for which the connection between the surface energy W and the form of the surface ∂V^f is found from the equation $-p^f + p^f_d = \delta W/\delta\eta$, where p^f_d is the fluid pressure in the region of its unknown surface. In this case the Hamiltonian is equal to the sum of (2.5) and W. The simplest example is the Hamiltonian formalism for potential motions in a fluid layer under an elastic plate [13].

3. We will consider the Lagrangian approach to investigate the potential motions of an ideal fluid. We will assume that at the initial instant of time $t = 0$, the coordinate axes of the Lagrangian system ξ^i, $i = 1, 2, 3$ coincide with the axes x^i of the Euler system. Each fluid particle will then have its Lagrangian coordinates ξ^i and the displacement vector $u^i(\xi^1, \xi^2, \xi^3, t)$. For vortex-free motions the components of the velocity of the fluid particles $v^j = \mathring{g}^{ij}\partial\varphi/\partial\xi^i$, where $\mathring{g}^{ij}$ is the metric tensor of the Lagrangian system when $t = 0$ [14].

We will choose the Euler system of coordinates x^i in such a way that its basis vectors at points traversed by the free surface of the fluid coincide with the basis vectors of the Lagrangian system ξ^i at the same points. In this case we have for the canonical variable

$$\eta = \int_{t_0}^{t} \sqrt{g}\, v^3 dt \tag{3.1}$$

Here g is the determinant of the metric tensor $g^{ij}(\xi^1, \xi^2, \xi^3, t)$ of the Lagrangian system on the free surface of the fluid. The quantity $\eta d\xi^1 d\xi^2$ is an elementary volume traversed by the fluid particles lying on its free surface in a time t.

The Hamiltonian equations in Lagrangian variables have the form (2.4), where η is defined by (3.1). The Hamiltonian is determined by the expressions

$$H = E + \int_{S^f} p^f u^3 \sqrt{\mathring{g}}\, d\xi^1 d\xi^2 - \int_{S^r} \frac{\partial u^3}{\partial t} \varphi^r \sqrt{\mathring{g}}\, d\xi^1 d\xi^2$$

$$E = T + \Pi, \quad T = \frac{1}{2}\int_V \mathring{g}^{ij} \frac{\partial\varphi}{\partial\xi^i}\frac{\partial\varphi}{\partial\xi^j} \sqrt{\mathring{g}}\, d\xi^1 d\xi^2 d\xi^3, \quad \Pi = \int_V U \sqrt{\mathring{g}}\, d\xi^1 d\xi^2 d\xi^3$$

The Lagrangian approach can be conveniently used when the regions $S^{f,r}$ in the Euler system of coordinates vary during the motion.

It was shown in [11, 12] that one can use the Lagrange equations to investigate the potential motions of an ideal fluid in the potential field of external forces, where one chooses as the generalized coordinates the normal displacements of the boundary of the fluid δx_n, and the Lagrange function L is equal to the difference between the kinetic and potential energy of the fluid. It can be seen that the equation $\delta\eta = \delta x_n$ is satisfied. Hence, the Hamiltonian approach considered here is equivalent to the Lagrangian approach and the function $\varphi^f = \delta L/\delta\eta$ is the generalized momentum.

4. To investigate the system of equations (2.4) when studying a large class of special motions when one of the Euler spatial variables varies over a limited range (a, b) and the functions $(\eta, \varphi^f) \in L_2(a, b)$, it is best to use orthogonal transformations with respect to this variable with weight $\sqrt{g}$. For example

$$\eta(x^1,x^2,t)=\sum_{m=0}^{\infty}\eta_m(x^1,t)f_m(x^2),\quad \varphi^f(x^1,x^2,t)=\sum_{m=0}^{\infty}\varphi_m(x^1,t)f_m(x^2) \tag{4.1}$$

$$\int_a^b f_m f_n\sqrt{g}\,dx^2=\delta_{mn}$$

In (4.1) the range (a, b) is the range of variation of the variable x^2 during the motion.

Substituting (4.1) into the expression for the variation of the Hamiltonian and integrating we obtain

$$\frac{\partial\eta_m}{\partial t}=\frac{\delta H}{\delta\varphi_m},\quad \frac{\partial\varphi_m}{\partial t}=-\frac{\delta H}{\delta\eta_m} \tag{4.2}$$

Henceforth we will use a system of coordinates where $\sqrt{g}=1$. If we choose functions of a trigonometric series in complex form as the orthogonal system, we have

$$\frac{\partial\eta_m}{\partial t}=\frac{\delta H}{\delta\varphi_m^*},\quad \frac{\partial\varphi_m}{\partial t}=-\frac{\delta H}{\delta\eta_m^*} \tag{4.3}$$

The asterisk denotes complex conjugation and m is the number of the harmonic.

When investigating localized motions in unbounded regions when $x^2\in(-\infty,\infty)$ it is best to use a Fourier integral transformation with respect to x^2. In this case, we obtain for the Fourier transforms F_η and F_φ of the functions η and φ^f

$$\frac{\partial F_\eta}{\partial t}=\frac{\delta H}{\delta F_m^*},\quad \frac{\partial F_\varphi}{\partial t}=-\frac{\delta H}{\delta F_\eta^*} \tag{4.4}$$

Equations of the form (4.3) were used in [15, 16] when investigating resonance excitation of periodic internal and surface waves in a fluid layer by a variable pressure on its surface. In this case the steady forced solutions correspond to extremal points of the Hamiltonian. The construction of these solutions reduces to an analysis of the solutions of an algebraic system of equations of infinite dimensions, which follow from (4.3) when $\partial\eta/\partial t=0$, $\partial\varphi f/\partial t=0$. It has been shown that solutions of this system for small-amplitude waves can be obtained in the form of asymptotic series. Knowing the value of the Hamiltonian at the initial given and extremal points one can draw conclusions regarding the possibility that the solution will reach a steady state from the initial state.

Equations (4.4) were used in [3, 6] to investigate the non-linear interaction between wave packets on the free surface of a uniform fluid layer.

We will consider the average Hamiltonian description of weakly modulated wave packets propagating on a uniform background, for which

$$\eta=\sum_m\eta_m(t,\mu\mathbf{x})\exp(i\mathbf{k}_m\mathbf{x}),\quad \varphi^f=\sum_m\varphi_m(t,\mu\mathbf{x})\exp(i\mathbf{k}_m\mathbf{x}) \tag{4.5}$$

$$\nu=(Lk_{\min})^{-1}\ll 1,\quad \mathbf{k}_m=(k_m^1,k_m^2),\quad \mathbf{x}=(x^1,x^2)$$

where μ is the dispersion parameter and L is the characteristic scale of spatial modulation. We will assume that $\mathbf{k}_m$ are all possible linear combinations of certain specified wave vectors $\mathbf{k}_1^0, \mathbf{k}_2^0, \ldots, \mathbf{k}_N^0$ and $k_{\min}=\min(|\mathbf{k}_1^0|, |\mathbf{k}_2^0|, \ldots, |\mathbf{k}_N^0|)$. The variables η_m are the slowly varying amplitudes of waves with wave vectors $\mathbf{k}_m$.

Substituting (4.5) into (2.5) we obtain

$$H=H_0+H_1 \tag{4.6}$$

$$H_0=\int_{F^f}\Omega(\mu\mathbf{x},t)dx^1dx^2,\quad H_1=\sum_m\int_{S^f}\Omega_m(\mu\mathbf{x},t)\exp(i\mathbf{k}_m\mathbf{x})dx^1dx^2$$

We will introduce the following notation for the mean value of the arbitrary function $\Phi(x)$

$$\Phi^l(\mathbf{x})=\underbrace{\int\int_{x^1}^{x^1+\Delta}\ldots\int}_{l}\underbrace{\int\int_{x^2}^{x^2+\Delta}\ldots\int}_{l}\Phi(\mathbf{x})\underbrace{dx^1\ldots dx^1}_{l}\underbrace{dx^2\ldots dx^2}_{l}/\Delta^{2l}$$

$$\Delta = k_{\min}^{-1}, \quad l = 0,\ 1,\ 2,\ldots$$

Substituting expressions (4.5) into (2.5), multiplying the equations obtained by $\exp(-ikx)$ and then calculating the mean values of both sides of the resulting equations, we obtain

$$\frac{\partial \eta_m^l}{\partial t} = \frac{\delta H_0}{\delta(\varphi_m^l)^*} + O(\mu^l), \quad \frac{\partial \varphi_m^l}{\partial t} = -\frac{\delta H_0}{\delta(\eta_m^l)^*} + O(\mu^l) \tag{4.7}$$

Further simplification of the system of equations (4.7) usually involves expanding the functions in powers of the small parameter ε, representing the non-linearity of the problem and proportional to the amplitudes of the wave packets. The coefficients of different powers of the non-linear terms in the asymptotic expansions depend on the metric of the system of coordinates employed and the form of the known boundary of the liquid volume.

We wish to thank A. G. Kulikovskii and G. A. Alekseyev for their comments on the results of this paper.

This research was carried out with financial support from the Russian Fund for Fundamental Research (93-013-17355).

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Translated by R.C.G.